Math 352 HW. #5

Homework problems are taken from "Real Analysis" by N. L. Carothers. The problems are color coded to indicate level of difficulty. The color green indicates an elementary problem, which you should be able to solve effortlessly. Yellow means that the problem is somewhat harder. Red indicates that the problem is hard. You should attempt the hard problems especially.

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

1. If $f \in BV[a,\ b]$ and $[c,\ d] \subset [a,\ b]$, show that $f \in BV[c,\ d]$ and $V_c^d f \le V_a^b f$.

2. Suppose that $f \in B[a,\ b]$. If $V_{a+\varepsilon}^b f \le M$ for all $\varepsilon > 0$, does it follow that *f* is of bounded variation on [*a*, *b*]? Is $V_a^b f \le M$? If not, what additional hypothesis on *f* would make this so?

3. If *f* is a polygonal function on [*a*, *b*], or if *f* is a polynomial, show that $V_a^b f = \int_a^b |f'(t)|\,dt$. This at least partly justifies our earlier claim that $V_a^b f$ behaves like an integral.

4. Suppose that $f_n \to f$ pointwise on [*a*, *b*]. If each f_n is increasing, show that *f* is increasing. If each f_n is of bounded variation, does it follow that *f* is of bounded variation? Explain.

5. If $f_n \to f$ pointwise on [*a*, *b*], show that $V(f_n,\ P) \to V(f,\ P)$ for any partition *P* of [*a*, *b*]. In particular, if we also have $V_a^b f_n \le K$ for all *n*, then $V_a^b f \le K$ too.

6. Here is a variation on Exercise 5: If $\{f_n\}$ is a sequence in *BV*[*a*, *b*], and if $f_n \to f$ pointwise on [*a*, *b*], show that $V_a^b f \le \liminf_{n\to\infty} V_a^b f_n$.

7. Given a sequence of scalars $\{c_n\}$ and a sequence of distinct points $\{x_n\}$ in (*a*, *b*), define $f(x) = c_n$ if $x = x_n$ for some *n*, and *f*(x) = 0 otherwise. Under what condition(s) is *f* of bounded variation on [*a*, *b*]?

8. Let I(x) = 0 if x < 0 and I(x) = 1 if $x \geq 0$. Given a sequence of scalars $\{c_n\}$ with $\sum_{n=1}^{\infty} |c_n| < \infty$ and a sequence of distinct points $\{x_n\}$ in (a, b], define $f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$ for $x \in [a,\ b]$. Show that $f \in BV[a,\ b]$ and that $V_a^b f = \sum_{n=1}^{\infty} |c_n|$.

9. Show that $f \in C[a,\ b] \cap BV[a,\ b]$ if and only of f can be written as the difference of two *strictly* increasing continuous functions.

10. Given $f \in BV[a,\ b]$, define $g(x) = f(x+)$ for $a \leq x < b$ and $g(b) = f(b)$. Prove that g is right continuous and of bounded variation on [a, b].